# VOLUMES OF NONNEGATIVE POLYNOMIALS, SUMS OF SQUARES AND POWERS OF LINEAR FORMS

GRIGORIY BLEKHERMAN

Abstract. We study the quantitative relationship between the cones of nonnegative polynomials, cones of sums of squares and cones of sums of powers of linear forms. We derive bounds on the volumes (raised to the power reciprocal to the ambient dimension) of compact sections of the three cones. We show that the bounds are asymptotically exact if the degree is fixed and number of variables tends to infinity. When the degree is larger than two it follows that there are significantly more non-negative polynomials than sums of squares and there are significantly more sums of squares than sums of powers of linear forms. Moreover, we quantify the exact discrepancy between the cones; from our bounds it follows that the discrepancy grows as the number of variables increases.

## 1. Introduction

Let $P_{n,2k}$ be the vector space of real homogeneous polynomials in $n$ variables of degree $2k$. There are three interesting convex cones in $P_{n,2k}$: The cone of nonnegative polynomials, $C = C_{n,2k}$

$$C = \left\{ f \in P_{n,2k} \mid f(x) \geq 0 \quad \text{for all} \quad x \in \mathbb{R}^n \right\}.$$

The cone of sums of squares, $Sq = Sq_{n,2k}$

$$Sq = \left\{ f \in P_{n,2k} \;\middle|\; f = \sum_i f_i^2 \quad \text{for some} \quad f_i \in P_{n,k} \right\}.$$

The cone of sums of $2k$-th powers of linear forms, $Lf = Lf_{n,2k}$

$$Lf = \left\{ f \in P_{n,2k} \;\middle|\; f = \sum_i l_i^{2k} \quad \text{for some linear forms} \quad l_i \in P_{n,1} \right\}.$$

A different notation of $P_{n,2k}$, $\Sigma_{n,2k}$ and $Q_{n,2k}$ respectively was employed by Reznick in the study of these cones [12]. The cones are clearly nested:

$$Lf_{n,2k} \subseteq Sq_{n,2k} \subseteq C_{n,2k}.$$

It is known that for quadratic forms these cones coincide. Moreover, it is not hard to show that in all other cases there are sums of squares

that are not $2k$-th powers of linear forms. Hilbert proved that in the cases $n = 2$, $k = 1$ and, $n = 3$ and $k = 2$, a nonnegative polynomial is necessarily a sum of squares; in all other cases there exist nonnegative polynomials that are not sums of squares [7]. The situation with respect to containment has therefore been completely known for a long time.

There remains, however, the question of the quantitative relationship between these cones. There are several known families of polynomials that are not sums of squares [4], [14]; however all of these examples lie close to the boundary of the cone of nonnegative polynomials. To the author's knowledge little except for the equality in the case of quadratic forms is known. In this paper we show that the picture is quite different for a fixed degree greater than 2.

For a convex set $K$ a good measure of size of $K$ that takes into account the effect of large dimensions is the volume of $K$ raised to the power reciprocal to the ambient dimension:

$$(\operatorname{Vol} K)^{1/\dim K}.$$

For example, homothetically expanding $K$ by a constant factor leads to an increase by the same factor in this normed volume.

We derive bounds on volumes, raised to the power reciprocal to the ambient dimension, of sections of the three cones with the hyperplane of all forms of integral 1 on the unit sphere $S^{n-1}$ in $\mathbb{R}^n$. We show that the bounds are asymptotically tight if the degree is fixed and number of variables tends to infinity. If the degree is greater than 2 then the order of dependence on the number of variables $n$ is quite different for the three cones. We remark that this indeed shows that asymptotically the cones differ drastically in size. These bounds provide us with the complete picture of metric dependence of the size of all three cones on the number of variables, when the degree is fixed.

We would also like to mention that the bounds that separate the cone of nonnegative polynomials from the cone of sums of squares are interesting from the point of view of computational complexity [16]. Namely, they show that it is not feasible in general to replace testing for positivity with testing whether a polynomial is a sum of squares, since for degree greater than two the sizes of the cones are drastically different. Some of the bounds given in this paper have already been proved by the author in [3]; we reproduce their proofs for the sake of completeness.

## 2. Main Theorems

We begin by introducing some notation. In order to compare the cones we take compact bases. Let $M = M_{n,2k}$ be the hyperplane of all

forms in $P_{n,2k}$ with integral 0 on the unit sphere $S^{n-1}$:

$$M_{n,2k} = \left\{ f \in P_{n,2k} \;\middle|\; \int_{S^{n-1}} f \, d\sigma = 0 \right\}.$$

Let $r^{2k}$ in $P_{n,2k}$ be the polynomial constant on the unit sphere $S^{n-1}$:

$$r^{2k} = (x_1^2 + \ldots + x_n^2)^k.$$

Let $M'$ be the affine hyperplane of all forms of integral 1 on the unit sphere $S^{n-1}$. We define compact convex bodies $\widetilde{C}$, $\widetilde{Sq}$ and $\widetilde{Lf}$ by intersecting the respective cones with $M'$ and then translating the compact intersection into $M$ by subtracting $r^{2k}$. Formally we can define $\widetilde{C}$, $\widetilde{Sq}$ and $\widetilde{Lf}$ as the sets of all forms $f$ in $M_{n,2k}$ such that $f + r^{2k}$ lies in the respective cone:

$$\begin{aligned} \widetilde{C} &= \{ f \in M_{n,2k} \quad | \quad f + r^{2k} \in C \}, \\ \widetilde{Sq} &= \{ f \in M_{n,2k} \quad | \quad f + r^{2k} \in Sq \}, \\ \widetilde{Lf} &= \{ f \in M_{n,2k} \quad | \quad f + r^{2k} \in Lf \}. \end{aligned}$$

We note that these sections are the natural ones to take since $M_{n,2k}$ is the only linear hyperplane in $P_{n,2k}$ that is preserved by an orthogonal change of coordinates in $\mathbb{R}^n$.

We work with the following Euclidean metric on $P_{n,2k}$, which we call the integral or $L^2$ metric,

$$\langle f \, , g \rangle = \int_{S^{n-1}} f g \, d\sigma,$$

where $\sigma$ is the rotation invariant probability measure on $S^{n-1}$. We use $D_M$ to denote the dimension of $M_{n,2k}$, $S_M$ to denote the unit sphere in $M_{n,2k}$ and $B_M$ to denote the unit ball in $M_{n,2k}$. The main results of this paper are the following three theorems:

**Theorem 2.1.** *There exist constants $\alpha_1$ and $\beta_1 > 0$ dependent only on $k$ such that*

$$\beta_1 n^{-1/2} \leq \left( \frac{Vol \, \widetilde{C}}{Vol \, B_M} \right)^{1/D_M} \leq \alpha_1 n^{-1/2}.$$

**Theorem 2.2.** *There exist constants $\alpha_2$ and $\beta_2 > 0$ dependent only on $k$ such that*

$$\beta_2 n^{-k/2} \left( \frac{Vol \, \widetilde{Sq}}{Vol \, B_M} \right)^{1/D_M} \leq \alpha_2 n^{-k/2}.$$

**Theorem 2.3.** *There exist constants $\alpha_3$ and $\beta_3 > 0$ dependent only on $k$ such that for all $\epsilon > 0$ and $n$ large enough*

$$\beta_3 n^{-k+1/2} \leq \left( \frac{Vol\,\widetilde{Lf}}{Vol\,B_M} \right)^{1/D_M} \leq \alpha_3 n^{-k+1/2+\epsilon}.$$

We observe that if the degree $2k$ is equal to two, then all of the above bounds agree asymptotically. However if the degree is greater than two then we see that the bases $\widetilde{C}$, $\widetilde{Sq}$ and $\widetilde{Lf}$ asymptotically have quite different volumes.

The rest of the paper is structured as follows. In Section 3 we collect preliminary material necessary for the proofs. Since many of the estimates used are technical in nature, in Section 4 we give an outline of the proofs postponing the technical details for the later sections. In Section 5 we prove the bounds for the cone of nonnegative polynomials. In Section 6 we introduce a different metric on $P_{n,2k}$ and prove duality results used later on. In Section 7 we prove the bounds for the cone of sums of squares and in Section 8 we prove the bounds for the cone of sums of powers of linear forms.

## 3. Preliminaries

### 3.1. The Action of the Orthogonal Group on $P_{n,2k}$.

There is the following action of $SO(n)$ on $P_{n,2k}$,

$$A \in SO(n) \quad \text{sends} \quad f \in P_{n,2k} \quad \text{to} \quad Af = f(A^{-1}x).$$

We observe that the cones $C$, $Sq$ and $Lf$ are invariant under this action and so is $M_{n,2k}$, the hyperplane of polynomials of integral 0. Therefore the sections $\widetilde{C}$, $\widetilde{Sq}$ and $\widetilde{Lf}$ are fixed by $SO(n)$ as well.

Let $\Delta$ be the Laplace differential operator:

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}.$$

A form $f$ such that

$$\Delta(f) = 0,$$

is called *harmonic*. We will need the fact that the irreducible components of this representation are subspaces $H_{n,2l}$ for $0 \leq l \leq k$, which have the following form:

$$H_{n,2l} = \left\{ f \in P_{n,2k} \mid f = r^{2k-2l} h \quad \text{where} \quad h \in P_{n,2l} \quad \text{is harmonic} \right\}.$$

For $v \in \mathbb{R}^n$ the functional

$$\lambda_v : M_{n,2k} \longrightarrow \mathbb{R}, \qquad \lambda_v(f) = f(v),$$

is linear and therefore there exists a form $q_v \in M$ such that

$$\lambda_v(f) = \langle q_v\,, f\rangle.$$

There are explicit descriptions of the polynomials $q_v$, under a suitable normalization they are so called Gegenbauer or ultraspherical polynomials. We will only need the property that for $v \in S^{n-1}$

$$||\, q_v||_2 = \sqrt{D_M}.$$

For more details on this representation of $SO(n)$ see [17].

3.2. **The Blaschke-Santaló Inequality.**

Let $K$ be a full-dimensional convex body in $\mathbb{R}^n$ with origin in its interior and let $\langle\ ,\ \rangle$ be an inner product. We will use $K^\circ$ to denote the polar of $K$,

$$K^\circ = \big\{x \in \mathbb{R}^n \ \mid\ \langle x\,, y\rangle \leq 1 \quad \text{for all} \quad y \in K\big\}.$$

Now suppose that a point $z$ is in the interior of $K$ and let $K^z$ be the polar of $K$ when $z$ is translated to the origin:

$$K^z = \big\{x \in \mathbb{R}^n \ \mid\ \langle x - z\,, y - z\rangle \leq 1 \quad \text{for all} \quad y \in K\big\}.$$

The point $z$ at which the volume of $K^z$ is minimal is unique and it is called the Santaló point of $K$. Moreover the following inequality on volumes of $K$ and $K^z$ holds:

$$\frac{\operatorname{Vol} K \operatorname{Vol} K^z}{(\operatorname{Vol} B)^2} \leq 1,$$

where $B$ is the unit ball of $\langle\ ,\ \rangle$ and $z$ is the Santaló point of $K$. This is known as the Blaschke-Santaló inequality [9].

## 4. Outline of Proofs

Since many of the following proofs are technical we would like to first give an informal outline.

We begin with the description of the proofs for the cone of nonnegative polynomials. We observe that $\widetilde{C}$ is the convex body of forms of integral 0 on $S^{n-1}$, such that the minimum of the forms on $S^{n-1}$ is at least $-1$,

$$\widetilde{C} = \big\{f \in M_{n,2k} \quad \mid \quad f(x) \geq -1 \quad \text{for all} \quad x \in S^{n-1}\big\}.$$

Let $B_\infty$ be the unit ball of $L^\infty$ norm in $M_{n,2k}$,

$$B_\infty = \big\{f \in M_{n,2k} \quad \mid \quad |f(x)| \leq 1 \quad \text{for all} \quad x \in S^{n-1}\big\}.$$

It follows that

$$B_\infty = \widetilde{C} \cap -\widetilde{C} \qquad \text{and therefore} \qquad B_\infty \subset \widetilde{C}.$$

However, using the Blaschke-Santaló inequality and a theorem of Rogers and Shephard [10] we can show that conversely

$$\left(\frac{\operatorname{Vol} B_\infty}{\operatorname{Vol} \widetilde{C}}\right)^{1/D_M} \geq 1/4.$$

Therefore it suffices to derive upper and lower bounds for the volume of $B_\infty$.

For the lower bound we reduce the proof to bounding the average $L^\infty$ norm of a form in $M_{n,2k}$,

$$\int_{S_M} ||f||_\infty \, d\mu,$$

where $S_M$ is the unit sphere in $M_{n,2k}$ and $\mu$ is the rotation invariant probability measure on $S_M$. The key idea is to estimate $||f||_\infty$ using $L^{2p}$ norms for some large $p$. An inequality of Barvinok [1] is used to see that taking $p = n$ suffices for $||f||_{2p}$ to be within a constant factor of $||f||_\infty$. The proof is completed with some estimates.

The techniques used for the proof of the upper bound are quite different. Let $\nabla f$ be the gradient of $f \in P_{n,2k}$,

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right),$$

and let $\langle \nabla f \,, \nabla f \rangle$ be the following polynomial giving the squared length of the gradient of $f$,

$$\langle \nabla f \,, \nabla f \rangle = \left(\frac{\partial f}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial f}{\partial x_n}\right)^2.$$

The key to the proof is the following theorem of Kellogg [8] which tells us that for homogeneous polynomials the maximum length of the gradient on the unit sphere $S^{n-1}$ is equal to the maximum absolute value of the polynomial on $S^{n-1}$ multiplied by the degree of the polynomial:

$$||\langle \nabla f \,, \nabla f \rangle||_\infty = 4k^2 ||f||_\infty^2.$$

Now we define a different inner product on $P_{n,2k}$ which we call the gradient inner product,

$$\langle f \,, g \rangle_G = \frac{1}{4k^2} \int_{S^{n-1}} \langle \nabla f \,, \nabla g \rangle \, d\sigma.$$

We denote the norm of $f$ in the gradient metric by $||f||_G$ and the unit ball of the gradient metric in $M_{n,2k}$ by $B_G$. We observe that

$$||f||_G = \frac{1}{4k^2} \int_{S^{n-1}} \langle \nabla f \,, \nabla f \rangle \, d\sigma,$$

and hence it follows that

$$||f||_G \leq ||f||_\infty \qquad \text{and therefore} \qquad B_\infty \subset B_G.$$

The relationship between the gradient metric and the integral metric can be calculated precisely by using the fact that both metrics are $SO(n)$-invariant. Therefore these metrics are constant multiples of each other in the irreducible subspaces of the $SO(n)$ representation and the constants can be calculated directly using the Stokes' formula. Hence we obtain an upper bound for the volume of $B_\infty$ in terms of the volume of $B_M$, the unit ball of the $L^2$ metric in $M_{n,2k}$.

The intuitive idea of the proof is as follows. In the $L^2$ metric we have,

$$||f||_2 \leq ||f||_\infty \qquad \text{and therefore} \qquad B_\infty \subset B_M.$$

However we give up too much in this estimate. On the other hand, it is not hard to show that

$$f^2(x) \leq 4k^2 \langle \nabla f \,, \nabla f \rangle \quad \text{for all} \quad x \in S^{n-1}.$$

Direct computations show that using the gradient metric gives us a better estimate and that this estimate is fine enough for our purposes.

The proof of the upper bound for the cone of sums of squares is quite similar to the proof of the lower bound for the cone of nonnegative polynomials. We define the following norm on $P_{n,2k}$,

$$||f||_{sq} = \max_{g \in S_{P_{n,k}}} |\langle f \,, g^2 \rangle|,$$

where $S_{P_{n,k}}$ is the unit sphere in $P_{n,k}$. Using inequalities from convexity we can reduce the proof to bounding the average $||f||_{sq}$.

To every form $f \in P_{n,2k}$ we can associate a quadratic form $H_f$ on $P_{n,2k}$ by letting

$$H_f(g) = \langle f \,, g^2 \rangle \qquad \text{for} \qquad g \in P_{n,k}.$$

It follows that

$$||f||_{sq} = ||H_f||_\infty.$$

Now we can estimate $||H_f||_\infty$ by high $L^{2p}$ norms of $H_f$ and the proof is finished using similar ideas to the proof for the case of nonnegative polynomials.

For the remainder of the proofs we will need to consider yet another metric on $P_{n,2k}$. To a form $f \in P_{n,2k}$,

$$f = \sum_{\alpha=(i_1,\ldots,i_n)} c_\alpha x_1^{i_1} \ldots x_n^{i_n}.$$

we formally associate the differential operator $D_f$:

$$D_f = \sum_{\alpha=(i_1,\dots,i_n)} c_\alpha \frac{\partial^{i_1}}{\partial x_1^{i_1}} \cdots \frac{\partial^{i_n}}{\partial x_n^{i_n}}.$$

We define the following metric on $P_{n,2k}$, which we call the differential metric:

$$\langle f\,, g\rangle_D = D_f(g).$$

It is not hard to check that this indeed defines a symmetric positive definite bilinear form, which is invariant under the action of $SO(n)$. The relationship between the differential metric and the integral metric can be calculated precisely.

For the proof of the lower bound for the cone of sums of squares we show that the dual cone $Sq_d^*$ of $Sq$ with respect to the differential metric is contained in $Sq$. Therefore we can derive a lower bound on the volume of $\widetilde{Sq}$ by using the Blaschke-Santaló inequality.

It can be shown that the cone of sums of $2k$-th powers of linear forms $Lf$ is dual to $C$ in the differential metric. The proofs of the bounds follow from the bounds derived for $\widetilde{C}$ and the Blaschke-Santaló inequality.

## 5. Nonnegative Polynomials

In this section we prove Theorem 2.1. Here is the precise statement of the bounds:

**Theorem 5.1.** *There are the following bounds on the volume of $\widetilde{C}$:*

$$\frac{1}{2\sqrt{4k+2}}\, n^{-1/2} \leq \left(\frac{Vol\,\widetilde{C}}{Vol\,B_M}\right)^{1/D_M} \leq 4\left(\frac{2k^2}{4k^2+n-2}\right)^{1/2}.$$

### 5.1. **Proof of the Lower Bound.**

For a real Euclidean vector space $V$ with the unit sphere $S_V$ and a function $f : V \to \mathbb{R}$ we use $||f||_p$ to denote the $L^p$ norm of $f$:

$$||f||_p = \left(\int_{S_V} |f|^p \, d\mu\right)^{1/p} \qquad \text{and} \qquad ||f||_\infty = \max_{x\in S_V} |f(x)|.$$

We begin by observing that $\widetilde{C}$ is a convex body in $M_{n,2k}$ with origin in its interior and the boundary of $\widetilde{C}$ consists of polynomials with minimum $-1$ on $S^{n-1}$. Therefore the gauge $G_C$ of $\widetilde{C}$ is given by:

$$G_C(f) = |\min_{v\in S^{n-1}} f(v)\,|.$$

By using integration in polar coordinates in $M$ we obtain the following expression for the volume of $\widetilde{C}$,

$$\left(\frac{\operatorname{Vol}\widetilde{C}}{\operatorname{Vol}\,B_M}\right)^{\frac{1}{D_M}} = \left(\int_{S_M} G_C^{-D_M}\, d\mu\right)^{\frac{1}{D_M}}, \tag{5.1.1}$$

where $\mu$ is the rotation invariant probability measure on $S_M$. The relationship (5.1.1) holds for any convex body with origin in its interior [11, p. 91].

We interpret the right hand side of (5.1.1) as $||G_C^{-1}||_{D_M}$, and by Hölder's inequality

$$||G_C^{-1}||_{D_M} \geq ||G_C^{-1}||_1.$$

Thus,

$$\left(\frac{\operatorname{Vol}\widetilde{C}}{\operatorname{Vol}B_M}\right)^{\frac{1}{D_M}} \geq \int_{S_M} G_C^{-1}\, d\mu.$$

By applying Jensen's inequality [6, p.150], with convex function $y = 1/x$ it follows that,

$$\int_{S_M} G_C^{-1}\, d\mu \geq \left(\int_{S_M} G_C\, d\mu\right)^{-1}.$$

Hence we see that

$$\left(\frac{\operatorname{Vol}\widetilde{C}}{\operatorname{Vol}B_M}\right)^{\frac{1}{D_M}} \geq \left(\int_{S_M} |\min f|\, d\mu\right)^{-1}.$$

Clearly, for all $f \in P_{n,2k}$

$$||f||_\infty \geq |\min f|.$$

Therefore,

$$\left(\frac{\operatorname{Vol}\widetilde{C}}{\operatorname{Vol}B_M}\right)^{\frac{1}{D_M}} \geq \left(\int_{S_M} ||f||_\infty\, d\mu\right)^{-1}.$$

The proof of the lower bound of Theorem 5.1 is now completed by the following estimate.

**Theorem 5.2.** *Let $S_M$ be the unit sphere in $M_{n,2k}$ and let $\mu$ be the rotation invariant probability measure on $S_M$. Then the following inequality for the average $L^\infty$ norm over $S_M$ holds:*

$$\int_{S_M} ||f||_\infty\, d\mu \leq 2\sqrt{2n(2k+1)}.$$

*Proof.* It was shown by Barvinok in [1] that for all $f \in P_{n,2k}$,

$$||f||_\infty \leq \binom{2kn+n-1}{2kn}^{\frac{1}{2n}} ||f||_{2n}.$$

By applying Stirling's formula we can easily obtain the bound

$$\binom{2kn+n-1}{2kn}^{\frac{1}{2n}} \leq 2\sqrt{2k+1}.$$

Therefore it suffices to estimate the average $L^{2n}$ norm, which we denote by $A$:

$$A = \int_{S_M} ||f||_{2n} \, d\mu.$$

Applying Hölder's inequality we observe that

$$A = \int_{S_M} \left( \int_{S^{n-1}} f^{2n}(x) \, d\sigma \right)^{\frac{1}{2n}} d\mu \leq \left( \int_{S_M} \int_{S^{n-1}} f^{2n}(x) \, d\sigma \, d\mu \right)^{\frac{1}{2n}}.$$

By interchanging the order of integration we obtain

$$A \leq \left( \int_{S^{n-1}} \int_{S_M} f^{2n}(x) \, d\mu \, d\sigma \right)^{\frac{1}{2n}}. \tag{5.2.1}$$

We now note that by symmetry of $M$

$$\int_{S_M} f^{2n}(x) \, d\mu,$$

is the same for all $x \in S^{n-1}$. Therefore we see that in (5.2.1) the outer integral is redundant and thus

$$A \leq \left( \int_{S_M} f^{2n}(v) \, d\mu \right)^{\frac{1}{2n}}, \qquad \text{where } v \text{ is any vector in } S^{n-1}. \tag{5.2.2}$$

We recall from Section 3 that for $v \in S^{n-1}$ there there exists a form $q_v$ in $M$ such that

$$\langle f \, , q_v \rangle = f(v) \quad \text{for all} \quad f \in M \qquad \text{and} \qquad ||q_v||_2 = \sqrt{D_M}.$$

Rewriting (5.2.2) we see that

$$A \leq \left( \int_{S_M} \langle f \, , q_v \rangle^{2n} \, d\mu \right)^{\frac{1}{2n}}. \tag{5.2.3}$$

We observe that

$$\int_{S_M} \langle f \, , q_v \rangle^{2n} \, d\mu = (D_M)^n \, \frac{\Gamma(n+\frac{1}{2}) \, \Gamma(\frac{1}{2} D_M)}{\sqrt{\pi} \, \Gamma(\frac{1}{2} D_M + n)}.$$

We substitute this into (5.2.3) to obtain,

$$A \le \left( (D_M)^n \, \frac{\Gamma(n+\frac{1}{2}\,)\,\Gamma(\frac{1}{2}D_M)}{\sqrt{\pi}\,\Gamma(\frac{1}{2}D_M+n)} \right)^{\frac{1}{2n}}.$$

Since

$$\left(\frac{\Gamma(\frac{1}{2}D_M)}{\Gamma(\,\frac{1}{2}D_M+n)}\right)^{\frac{1}{2n}} \le \sqrt{\frac{2}{D_M}} \qquad \text{and} \qquad \left(\frac{\Gamma(n+1/2\,)}{\sqrt{\pi}}\right)^{\frac{1}{2n}} \le n^{1/2},$$

we see that

$$A \le (2n)^{1/2}.$$

The theorem now follows. $\square$

5.2. **Proof of the Upper Bound.**

We begin by noting that the origin is the only point in $M$ fixed by $SO(n)$. Let $\widetilde{C}^\circ$ be the polar of $\widetilde{C}$ in $M_{n,2k}$,

$$\widetilde{C}^\circ = \{f \in M_{n,2k} \mid \langle f\,, g\rangle \le 1 \quad \text{for all} \quad g \in \widetilde{C}\}.$$

Since $\widetilde{C}$ is fixed by the action of $SO(n)$ and Santaló point of a convex body is unique, it follows that the origin is the Santaló point of $\widetilde{C}$. We now use Blaschke-Santaló inequality, which applied to $\widetilde{C}$ gives us:

$$(\mathrm{Vol}\,\widetilde{C})\,(\mathrm{Vol}\,\widetilde{C}^\circ) \le (\mathrm{Vol}\,B_M)^2.$$

Therefore it would suffice to show that

$$\left(\frac{\mathrm{Vol}\,\widetilde{C}^\circ}{\mathrm{Vol}\,B_M}\right)^{1/D_M} \ge \frac{1}{4}\left(\frac{4k^2+n-2}{2k^2}\right)^{1/2}. \tag{5.2.4}$$

Let $B_\infty$ be the unit ball of the $L^\infty$ metric in $M_{n,2k}$,

$$B_\infty = \{f \in M \;\mid\; ||f||_\infty \le 1\}.$$

We observe that $B_\infty$ is clearly the intersection of $\widetilde{C}$ with $-\widetilde{C}$:

$$B_\infty = \widetilde{C} \cap -\widetilde{C}.$$

By taking polars it follows that

$$B_\infty^\circ = \mathrm{ConvexHull}\{C^\circ, -\,C^\circ\} \subset \widetilde{C}^\circ \oplus (-\,\widetilde{C}^\circ),$$

where $\oplus$ denotes Minkowski addition. By theorem of Rogers and Shephard, [10] p. 78, it follows that

$$\mathrm{Vol}\,B_\infty^\circ \le \binom{2D_M}{D_M}\mathrm{Vol}\,\widetilde{C}^\circ.$$

Since

$$\binom{2D_M}{D_M} \le 4^{D_M},$$

we obtain

$$\left(\frac{\mathrm{Vol}\,\widetilde{C}^\circ}{\mathrm{Vol}\,B_\infty^\circ}\right)^{1/D_M} \geq \frac{1}{4}.$$

Combining with (5.2.4) we see that we have reduced the lower bound of Theorem 5.1 to showing that

$$\left(\frac{\mathrm{Vol}\,B_\infty^\circ}{\mathrm{Vol}\,B_M}\right)^{1/D_M} \geq \left(\frac{4k^2+n-2}{2k^2}\right)^{1/2} \tag{5.2.5}$$

For a form $f$ we use $\nabla f$ to denote the gradient of $f$:

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right).$$

We also define a different Euclidean metric on $P_{n,2k}$ which we call the gradient metric:

$$\langle f\,, g\rangle_G = \frac{1}{4k^2}\int_{S^{n-1}} \langle \nabla f\,, \nabla g\rangle \, d\sigma.$$

We denote the unit ball in this metric by $B_G$ and the norm of $f$ by $||f||_G$. For $f \in P_{n,2k}$ let $\langle \nabla f\,, \nabla f\rangle$ be the following polynomial:

$$\langle \nabla f\,, \nabla f\rangle = \left(\frac{\partial f}{\partial x_1}\right)^2 + \dots + \left(\frac{\partial f}{\partial x_n}\right)^2.$$

It was shown by Kellogg in [8] that

$$||\langle \nabla f\,, \nabla f\rangle||_\infty = 4k^2||f||_\infty^2.$$

It clearly follows that

$$||f||_\infty \geq ||f||_G,$$

and therefore

$$B_\infty \subseteq B_G.$$

Polarity reverses inclusion and thus we see that

$$B_G^\circ \subseteq B_\infty^\circ \quad \text{and} \quad \mathrm{Vol}\,B_G^\circ = \frac{(\mathrm{Vol}\,B_M)^2}{\mathrm{Vol}\,B_G},$$

since $B_G$ is an ellipsoid. Thus (5.2.5) and consequently the upper bound of Theorem 5.1 will follow from the following lemma.

**Lemma 5.3.**

$$\left(\frac{Vol\,B_M}{Vol\,B_G}\right)^{1/D_M} \geq \left(\frac{4k^2+n-2}{2k^2}\right)^{1/2}.$$

*Proof.* It will suffice to show that for all $f \in M$

$$\langle f\,, f\rangle_G \geq \frac{4k^2+n-2}{2k^2}\langle f\,, f\rangle. \tag{5.3.1}$$

By the invariance of both inner products under the action of $SO(n)$, it is enough to prove (5.3.1) in the irreducible components of the representation.

First let $f$ be a harmonic form of degree $2d$ in $n$ variables. Then we claim that

$$\langle f\,, f\rangle = \frac{2d}{4d+n-2}\,\langle f\,, f\rangle_G.$$

Indeed consider the vector field $F = f(v)\nabla f$ on $S^{n-1}$. By the Divergence Theorem:

$$\int_{S^{n-1}} \langle F\,, v\rangle\, dx(v) = \int_{||x||\leq 1} \operatorname{div} F\, dx,$$

where $dx$ is the Lebesgue measure and $\operatorname{div} F$ is the divergence of $F$:

$$\operatorname{div} F = \frac{\partial F_1}{\partial x_1} + \ldots + \frac{\partial F_n}{\partial x_n}.$$

Since $f$ is homogeneous of degree $2d$, it follows that

$$\langle \nabla f\,, v\rangle = 2d\, f(v).$$

Therefore

$$\int_{S^{n-1}} \langle F\,, v\rangle\, dx = 2\omega_n d \int_{S^{n-1}} f^2\, d\sigma = 2\omega_n d\langle f\,, f\rangle,$$

where $\omega_n$ is the surface area of $S^{n-1}$. Since $f$ is harmonic it follows that

$$\operatorname{div} F = \left(\frac{\partial f}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial f}{\partial x_n}\right)^2 = \langle \nabla f\,, \nabla f\rangle.$$

We observe that $\langle \nabla f\,, \nabla f\rangle$ is a homogeneous polynomial of degree $4d-2$ and therefore

$$\int_{||x||\leq 1} \langle \nabla f\,, \nabla f\rangle\, dx = \frac{\omega_n}{4d+n-2}\int_{S^{n-1}} \langle \nabla f\,, \nabla f\rangle\, d\sigma.$$

The claim now follows.

Now suppose that $f = hr^{2k-2d}$ where $h$ is a harmonic form of degree $2d \leq 2k$. It is easy to check that

$$\langle f\,, f\rangle_G = \frac{d^2}{k^2}\langle h\,, h\rangle_G + \frac{k^2-d^2}{k^2}\langle h\,, h\rangle.$$

We know that

$$\langle h\,, h\rangle_G = \frac{4d+n-2}{2d}\langle h\,, h\rangle \quad \text{and} \quad \langle f\,, f\rangle = \langle h\,, h\rangle.$$

Thus

$$\langle f\,,f\rangle_G = \frac{2k^2 + d(n-2) + 2d^2}{2k^2}\langle f\,,f\rangle.$$

Since $f \in M_{n,2k}$ we know that $1 \leq d \leq k$. The minimum clearly occurs when $d = 1$ and we see that

$$\langle f\,,f\rangle_G \leq \frac{4k^2 + n - 2}{2k^2}\langle f\,,f\rangle.$$

The lemma now follows. □

## 6. The Differential Metric

Before we proceed with the proofs of Theorems 2.2 and 2.3 we will need some preparatory results that involve switching to a different Euclidean metric on $P_{n,2k}$.

To a form $f \in P_{n,2k}$,

$$f = \sum_{\alpha=(i_1,\ldots,i_n)} c_\alpha x_1^{i_1} \ldots x_n^{i_n}.$$

we formally associate the differential operator $D_f$:

$$D_f = \sum_{\alpha=(i_1,\ldots,i_n)} c_\alpha \frac{\partial^{i_1}}{\partial x_1^{i_1}} \cdots \frac{\partial^{i_n}}{\partial x_n^{i_n}}.$$

We define the following metric on $P_{n,2k}$, which we call the differential metric:

$$\langle f\,,g\rangle_D = D_f(g).$$

It is not hard to check that this indeed defines a symmetric positive definite bilinear form, which is invariant under the action of $SO(n)$. For a point $v \in S^{n-1}$ we will use $v^{2k}$ to denote the polynomial

$$v^{2k} = (v_1x_1 + \ldots + v_nx_n)^{2k}.$$

We also define an important linear operator $T : P_{n,2k} \to P_{n,2k}$, which to a form $f \in P_{n,2k}$ associates weighted average of forms $v^{2k}$ with the weight $f(v)$:

$$T(f) = \int_{S^{n-1}} f(v)v^{2k}\,d\sigma(v).$$

The operator $T$ was first introduced in a very different form by Reznick in [13]; we take our definition from [2]. The operator $T$ acts as a switch between our standard integral metric and the differential metric in the following sense:

**Lemma 6.1.** *The following identity relating the operator $T$ and the two metrics holds,*

$$\langle Tf\,,g\rangle_D = (2k)!\langle f\,,g\rangle.$$

*Proof.* We observe that

$$\langle Tf\,,g\rangle_D = \langle \int_{S^{n-1}} f(v)v^{2k}\,d\sigma(v)\,,g\rangle_D = \int_{S^{n-1}} \langle f(v)v^{2k}\,,g\rangle_D\,d\sigma(v).$$

Since

$$\langle v^{2k}\,,g\rangle_D = (2k)!g(v),$$

it follows that

$$\langle Tf\,,g\rangle_D = (2k)!\int_{S^{n-1}} f(v)g(v)\,d\sigma(v) = (2k)!\langle f\,,g\rangle.$$

□

Let $L$ be a full-dimensional cone in $P_{n,2k}$ such that $r^{2k}$ is in the interior of $L$ and $\int_{S^{n-1}} f\,d\sigma > 0$ for all non-zero $f$ in $L$. We define $\widetilde{L}$ as the set of all forms $f$ in $M$ such that $f + r^{2k}$ lies in $L$,

$$\widetilde{L} = \{f \in M \mid f + r^{2k} \in L\}.$$

We let $L_i^*$ be the dual cone of $L$ in the integral metric and $L_d^*$ be the dual cone of $L$ in the differential metric.

$$L_i^* = \{f \in P_{n,2k} \mid \langle f\,,g\rangle \geq 0 \quad \text{for all} \quad g \in L\},$$
$$L_d^* = \{f \in P_{n,2k} \mid \langle f\,,g\rangle_D \geq 0 \quad \text{for all} \quad g \in L\}.$$

We observe that $r^{2k}$ is in the interior of both $L_i^*$ and $L_d^*$ and also $\int_{S^{n-1}} f\,d\sigma > 0$ for all non-zero $f$ in both of the dual cones. Therefore we can similarly define $\widetilde{L_i^*}$ and $\widetilde{L_d^*}$ as sets of all forms $f$ in $M$ such that $f + r^{2k}$ lies in the respective cone.

**Lemma 6.2.** *Let $L$ be a full-dimensional cone in $P_{n,2k}$ such that $r^{2k}$ is the interior of $L$ and $\int_{S^{n-1}} f\,d\sigma > 0$ for all $f$ in $L$. Then there is the following relationship between the volumes of $\widetilde{L_i^*}$ and $\widetilde{L_d^*}$*

$$\frac{k!}{(n/2+2k)^k} \leq \left(\frac{Vol\,\widetilde{L_d^*}}{Vol\,\widetilde{L_i^*}}\right)^{1/D_M} \leq \left(\frac{k!}{(n/2+k)^k}\right)^{\alpha},$$

*where*

$$\alpha = 1 - \left(\frac{2k-1}{2k+n-2}\right)^2.$$

*Proof.* From Lemma 6.1 we see that

$$\langle f\,,g\rangle \geq 0 \quad \text{if and only if} \quad \langle Tf\,,g\rangle_D \geq 0 \quad \text{for all} \quad f,g \in P_{n,2k}.$$

Therefore it follows that $T$ maps $L_i^*$ to $L_d^*$,

$$T(L_i^*) = L_d^*.$$

It is hot hard to show that

$$T(r^{2k}) = cr^{2k} \quad \text{where} \quad c = \int_{S^{n-1}} x_1^{2k}\, d\sigma = \frac{\Gamma(\frac{2k+1}{2})\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n+2k}{2})}.$$

Therefore $\frac{1}{c}T$ fixes the hyperplane of all forms of integral 1 on the sphere and therefore $\frac{1}{c}T$ maps the section $\widetilde{L_i^*}$ to $\widetilde{L_d^*}$.

It is possible to describe precisely the action of $\frac{1}{c}T$ on $M_{n,2k}$, see [2]. It can be shown that $\frac{1}{c}T$ is a contraction operator and the exact coefficients of contraction can be computed. We only need the following estimate, which follows from [2] Lemma 7.4 by estimating the change in volume to be at most the largest contraction coefficient:

$$\left(\frac{\operatorname{Vol}\widetilde{L_d^*}}{\operatorname{Vol}\widetilde{L_i^*}}\right)^{1/D_M} \geq \frac{k!\Gamma(k+n/2)}{\Gamma(2k+n/2)}.$$

We observe that

$$\frac{k!\Gamma(k+n/2)}{\Gamma(2k+n/2)} \geq \frac{k!}{(n/2+2k)^k},$$

and therefore,

$$\left(\frac{\operatorname{Vol}\widetilde{L_d^*}}{\operatorname{Vol}\widetilde{L_i^*}}\right)^{1/D_M} \geq \frac{k!}{(n/2+2k)^k}.$$

Also from Lemma 7.4 of [2] it follows that contraction by the largest coefficient occurs in the space of all harmonic polynomials of degree $2k$ which has dimension

$$D_H = \binom{n+2k-1}{2k} - \binom{n+2k-3}{2k-2}.$$

Since the dimension of the ambient space $M$ is

$$D_M = \binom{n+2k-1}{2k} - 1,$$

we can estimate that

$$\frac{D_H}{D_M} \geq 1 - \left(\frac{2k-1}{n+2k-2}\right)^2.$$

Since we can also estimate the largest contraction coefficient from above,

$$\frac{k!\Gamma(k+n/2)}{\Gamma(2k+n/2)} \leq \frac{k!}{(n/2+k)^k},$$

the theorem now follows. $\square$

We also show the following theorem, which allows us to compare the cone of sums of squares to its dual.

**Lemma 6.3.** *The dual cone to the cone of sums of squares in the differential metric $Sq_d^*$ is contained in the cone of sums of squares $Sq$,*

$$Sq_d^* \subseteq Sq.$$

*Proof.* In this proof we will work exclusively with the differential metric on $P_{n,k}$ and $P_{n,2k}$. Let $W$ be the space of quadratic forms on $P_{n,k}$. For $A$, $B$ in $W$, with corresponding symmetric matrices $M_A$, $M_B$ the inner product of $A$ and $B$ is given by,

$$\langle A\,, B\rangle = \operatorname{tr} M_A M_B.$$

For $q \in P_{n,k}$ let $A_q$ be the rank one quadratic form giving the square of the inner product with $q$:

$$A_q(p) = \langle p\,, q\rangle_D^2.$$

Then for any $B \in W$

$$\langle A_q\,, B\rangle = B(q).$$

Now suppose $f \in Sq_d^*$. Let $H_f$ be the following quadratic form on $P_{n,k}$:

$$H_f(p) = \langle p\,, f^2\rangle_D.$$

Since $f \in Sq_d^*$, the quadratic form $H_f$ is clearly positive semidefinite. Therefore $H_f$ can be written as a nonnegative linear combination of forms of rank 1:

$$H_f = \sum A_q \qquad \text{for some} \qquad q \in P_{n,k}. \tag{6.3.1}$$

Let $V$ be the subspace of $W$ given by the linear span of the forms $H_f$ for all $f \in P_{n,2k}$. Let $\mathbb{P}$ be the operator of orthogonal projection onto $V$. We claim that

$$\mathbb{P}(A_q) = \binom{2k}{k}^{-1} H_{q^2}.$$

It suffices to show that $A_q - \binom{2k}{k}^{-1} H_{q^2}$ is orthogonal to the forms $H_{v^{2k}}$ since these forms span $V$. We observe that

$$H_{v^{2k}}(p) = (2k)!p(v)^{2k} = \frac{(2k)!A_{v^k}(p)}{(k!)^2} = \binom{2k}{k} A_{v^k}(p).$$

Therefore we see that

$$\langle A_q - \binom{2k}{k}^{-1} H_{q^2}\,, H_{v^{2k}}\rangle = H_{v^{2k}}(q) - \langle H_{q^2}\,, A_{v^k}\rangle = H_{v^{2k}}(q) - H_{q^2}(v^k) = 0.$$

Now we apply $\mathbb{P}$ to both sides of (6.3.1). It follows that

$$H_f = \mathbb{P}\Big(\sum A_q\Big) = \sum \binom{2k}{k}^{-1} H_{q^2} = \binom{2k}{k}^{-1} H_{\sum q^2}.$$

Therefore $f$ is a sum of squares. $\square$

## 7. Sums of Squares

In this section we prove Theorem 2.2. The full statement of the bounds is the following,

**Theorem 7.1.** *There are the following bounds for the volume of $\widetilde{Sq}$:*

$$\frac{(k!)^2}{4^{2k}(2k)!\sqrt{24}}\frac{n^{k/2}}{(n/2+2k)^k} \leq \left(\frac{Vol\,\widetilde{Sq}}{Vol\,B_M}\right)^{1/D_M} \leq \frac{4^{2k}(2k)!\sqrt{24}}{k!}\,n^{-k/2}.$$

### 7.1. **Proof of the Upper Bound.**

Let us begin by considering the support function of $\widetilde{Sq}$, which we call $L_{\widetilde{Sq}}$:

$$L_{\widetilde{Sq}}(f) = \max_{g\,\in\,\widetilde{Sq}} \langle f\,,g\rangle.$$

The average width $W_{\widetilde{Sq}}$ of $\widetilde{Sq}$ is given by

$$W_{\widetilde{Sq}} = 2\int_{S_M} L_{\widetilde{Sq}}\,d\mu.$$

We now recall Urysohn's Inequality [15, p.318] which applied to $\widetilde{Sq}$ gives

$$\left(\frac{\mathrm{Vol}\,\widetilde{Sq}}{\mathrm{Vol}\,B_M}\right)^{\frac{1}{D_M}} \leq \frac{W_{\widetilde{Sq}}}{2}. \tag{7.1.1}$$

Therefore it suffices to obtain an upper bound for $W_{\widetilde{Sq}}$.

Let $S_{P_{n,k}}$ denote the unit sphere in $P_{n,k}$. We observe that extreme points of $\widetilde{Sq}$ have the form

$$g^2 - r^{2k} \qquad \text{where} \qquad g \in P_{n,k} \qquad \text{and} \qquad \int_{S^{n-1}} g^2\,d\sigma = 1.$$

For $f \in M$,

$$\langle f\,,r^{2k}\rangle = \int_{S^{n-1}} f\,d\sigma = 0,$$

and therefore,

$$L_{\widetilde{Sq}}(f) = \max_{g\,\in S_{P_{n,k}}} \langle f\,,g^2\rangle.$$

We now introduce a norm on $P_{n,2k}$, which we denote $||\;||_{sq}$:

$$||f||_{sq} = \max_{g\,\in\,S_{P_{n,k}}} |\langle f\,,g^2\rangle|.$$

It is clear that

$$L_{Sq}(f) \leq ||f||_{Sq}.$$

Therefore by (7.1.1) it follows that

$$\left(\frac{\operatorname{Vol}\widetilde{Sq}}{\operatorname{Vol}B_M}\right)^{\frac{1}{D_M}} \leq \int_{S_M} ||f||_{sq}\, d\mu.$$

The proof of the upper bound of Theorem 7.1 is reduced to the estimate below.

**Theorem 7.2.** *There is the following bound for the average* $||\ ||_{sq}$ *over* $S_M$*:*

$$\int_{S_M} ||f||_{sq}\, d\mu \ \leq\ \frac{4^{2k}(2k)!\sqrt{24}}{k!}\, n^{-k/2}.$$

*Proof.* For $f \in P_{n,2k}$ we introduce a quadratic form $H_f$ on $P_{n,k}$:

$$H_f(g) = \langle f\,, g^2\rangle \qquad \text{for} \qquad g \in P_{n,k}.$$

We note that

$$||f||_{sq} = \max_{g\,\in\, S_{P_{n,k}}} |\langle f\,, g\rangle| = ||H_f||_\infty.$$

We bound $||H_f||_\infty$ by a high $L^{2p}$ norm of $H_f$. Since $H_f$ is a form of degree 2 on the vector space $P_{n,k}$ of dimension $D_{n,k}$ it follows by the inequality of Barvinok in [1] applied in the same way as in the proof of Theorem 2.1 that

$$||H_f||_\infty \leq 2\sqrt{3}\, ||H_f||_{2D_{n,k}}.$$

Therefore it suffices to estimate:

$$A = \int_{S_M} ||H_f||_{2D_{n,k}}\, d\mu = \int_{S_M}\left(\int_{S_{P_{n,k}}} \langle f\,, g^2\rangle^{\,2D_{n,k}}\, d\sigma(g)\, d\mu(f)\right)^{\frac{1}{2D_{n,k}}}.$$

We apply Hölder's inequality to see that

$$A \leq \left(\int_{S_M}\int_{S_{P_{n,k}}} \langle f\,, g^2\rangle^{\,2D_{n,k}}\, d\sigma(g)\, d\mu(f)\right)^{\frac{1}{2D_{n,k}}}.$$

By interchanging the order of integration we obtain

$$A \leq \left(\int_{S_{P_{n,k}}}\int_{S_M} \langle f\,, g^2\rangle^{\,2D_{n,k}}\, d\mu(f)\, d\sigma(g)\right)^{\frac{1}{2D_{n,k}}}. \tag{7.2.1}$$

Now we observe that the inner integral

$$\int_{S_M} \langle f\,, g^2\rangle^{\,2D_{n,k}}\, d\mu(f),$$

clearly depends only on the length of the projection of $g^2$ into $M$. Therefore we have

$$\int_{S_M} \langle f\,, g^2\rangle^{2D_{n,k}}\, d\mu(f) \le \; ||g^2||_2^{2D_{n,k}} \int_{S_M} \langle f\,, p\rangle^{2D_{n,k}}\, d\mu(f),$$

$$\text{for any} \quad p \in S_M.$$

We observe that

$$||g^2||_2 = (||g||_4)^2 \qquad \text{and} \qquad ||g||_2 = 1.$$

By a result of Duoandikoetxea [5] Corollary 3 it follows that

$$||g^2||_2 \le 4^{2k}.$$

Hence we obtain

$$\int_{S_M} \langle f\,, g^2\rangle^{2D_{n,k}}\, d\mu(f) \le 4^{4kD_{n,k}} \int_{S_V} \langle f\,, p\rangle^{2D_{n,k}}\, d\mu(f).$$

We note that this bound is independent of $g$ and substituting into (7.2.1) we get

$$A \le 4^{2k}\left(\int_{S_V} \langle f\,, p\rangle^{2D_{n,k}}\, d\mu(f)\right)^{\frac{1}{2D_{n,k}}}.$$

Since $p \in S_M$ we have

$$\int_{S_M} \langle f\,, p\rangle^{2D_{n,k}}\, d\mu(f) = \frac{\Gamma(D_{n,k}+\frac{1}{2})\Gamma(\frac{1}{2}D_M)}{\sqrt{\pi}\,\Gamma(D_{n,k}+\frac{1}{2}D_M)}.$$

We use the following easy inequalities:

$$\left(\frac{\Gamma(\frac{1}{2}D_M)}{\Gamma(D_{n,k}+\frac{1}{2}D_M)}\right)^{\frac{1}{2D_{n,k}}} \le \sqrt{\frac{2}{D_M}}$$

and

$$\left(\frac{\Gamma(D_{n,k}+\frac{1}{2})}{\sqrt{\pi}}\right)^{\frac{1}{2D_{n,k}}} \le \sqrt{D_{n,k}},$$

to see that

$$A \le 4^{2k}\sqrt{\frac{2D_{n,k}}{D_M}}.$$

We now recall that

$$D_{n,k} = \binom{n+k-1}{k} \qquad \text{and} \qquad D_M = \binom{n+2k-1}{2k} - 1.$$

Therefore

$$\sqrt{\frac{D_{n,k}}{D_M}} \le \frac{(2k)!}{k!}\, n^{-k/2}.$$

Thus

$$A \leq \frac{4^k (2k)! \sqrt{2}}{k!} n^{-k/2}.$$

The theorem now follows. $\square$

### 7.2. **Proof of the Lower Bound.**

We begin with a corollary of Theorem 7.2. Let $B_{sq}$ be the unit ball of the norm $||\ ||_{sq}$,

$$B_{sq} = \{f \in M \mid ||f||_{sq} \leq 1\}.$$

From Theorem 7.2 we know that

$$\int_{S_M} ||f||_{sq}\, d\mu \leq \frac{4^{2k}(2k)!\sqrt{24}}{k!} n^{-k/2}.$$

It follows in the same way as in the section 3.1 that

$$\left(\frac{\operatorname{Vol} B_{sq}}{\operatorname{Vol} B_M}\right)^{1/D_M} \geq \frac{k!}{4^{2k}(2k)!\sqrt{24}} n^{k/2}.$$

Now let $\widetilde{Sq}^{\circ}$ be the polar of $\widetilde{Sq}$ in $M$. It follows easily that $B_{sq}$ is the intersection of $\widetilde{Sq}^{\circ}$ and $-\widetilde{Sq}^{\circ}$.

$$B_{sq} = \widetilde{Sq}^{\circ} \cap -\widetilde{Sq}^{\circ}.$$

Let $Sq_i^*$ be the dual cone of $Sq$ in the integral metric and let $\widetilde{Sq_i^*}$ be defined in the same way as for the previous cones. It is not hard to check that $\widetilde{Sq}^{\circ}$ is the negative of $\widetilde{Sq_i^*}$,

$$\widetilde{Sq}^{\circ} = -\widetilde{Sq_i^*}.$$

Therefore we see that

$$\left(\frac{\operatorname{Vol} \widetilde{Sq_i^*}}{\operatorname{Vol} B_M}\right)^{1/D_M} \geq \frac{k!}{4^{2k}(2k)!\sqrt{24}} n^{k/2}.$$

Now we observe that $r^{2k}$ is in the interior of $Sq$ and also for all non-zero $f$ in $Sq$ we have $\int_{S^{n-1}} f\, d\sigma > 0$. Therefore we can apply Lemma 6.2 to $Sq$ and it follows that

$$\left(\frac{\operatorname{Vol} \widetilde{Sq_d^*}}{\operatorname{Vol} \widetilde{Sq_i^*}}\right)^{1/D_M} \geq \frac{k!}{(n/2+2k)^k}.$$

Combining with (7.2) we see that

$$\left(\frac{\operatorname{Vol} \widetilde{Sq_d^*}}{\operatorname{Vol} B_M}\right)^{1/D_M} \geq \frac{(k!)^2}{4^{2k}(2k)!\sqrt{24}} \frac{n^{k/2}}{(n/2+2k)^k}.$$

By Lemma 6.3 we know that $Sq_d^*$ in contained in $Sq$ and therefore

$$\widetilde{Sq_d^*} \subseteq \widetilde{Sq}.$$

The lower bound now follows.

## 8. Sums of 2k-th Powers of Linear Forms

In this section we prove Theorem 2.3. Here is the precise statement of the bounds,

**Theorem 8.1.** *There are the following bounds for the volume of $\widetilde{Lf}$:*

$$\frac{k!\sqrt{4k^2+n-2}}{4k\sqrt{2}(n/2+2k)^k} \leq \left(\frac{Vol\,\widetilde{Lf}}{Vol\,B_M}\right)^{1/D_M} \leq 2\sqrt{n(4k+2)}\left(\frac{k!}{(n/2+k)^k}\right)^{\alpha},$$

*where*

$$\alpha = 1 - \left(\frac{2k-1}{n+2k-2}\right)^2.$$

### 8.1. **Proof of the Lower Bound.**

We observe that the cone of sums of $2k$-th powers of linear forms is dual to the cone of nonnegative polynomials in the differential metric,

$$Lf = C_d^*,$$

since in the differential metric,

$$\langle f\,, v^{2k}\rangle_D = (2k)!f(v) \quad \text{for all} \quad f \in P_{n,2k}.$$

Therefore it follows that

$$\widetilde{Lf} = \widetilde{C_d^*}.$$

We first consider the dual cone $C_i^*$ of $C$ in the integral metric. Similarly to the situation with the cone of sums of squares it is not hard to check that the dual $\widetilde{C}^\circ$ of $\widetilde{C}$ in $M$ with respect to the integral metric is $-\widetilde{C_i^*}$,

$$\widetilde{C}^\circ = -\widetilde{C_i^*}.$$

We recall that in Section 3.2 we have shown (5.2.4):

$$\left(\frac{\mathrm{Vol}\,\widetilde{C}^\circ}{\mathrm{Vol}\,B_M}\right)^{1/D_M} \geq \frac{1}{4}\left(\frac{4k^2+n-2}{2k^2}\right)^{1/2}.$$

Since $C$ has $r^{2k}$ in its interior and $\int_{S^{n-1}} f\,d\sigma > 0$ for all non-zero $f$ in $C$, we can apply Lemma 6.2 to $C$ and we obtain,

$$\left(\frac{\mathrm{Vol}\,\widetilde{C_d^*}}{\mathrm{Vol}\,\widetilde{C_i^*}}\right)^{1/D_M} \geq \frac{k!}{(n/2+2k)^k}.$$

Since $\widetilde{Lf} = \widetilde{C_d^*}$ and $\widetilde{C}^\circ = -\widetilde{C_i^*}$ we can combine with (5.2.4) and we get:

$$\left(\frac{\operatorname{Vol}\widetilde{Lf}}{\operatorname{Vol}B_M}\right)^{1/D_M} \geq \frac{k!}{4k\sqrt{2}}\frac{(4k^2+n-2)^{1/2}}{(n/2+2k)^k}.$$

8.2. **Proof of the Upper Bound.**

We begin by applying the Blaschke-Santaló inequality to $\widetilde{C}$ as in Section 3.2 to obtain

$$\frac{\operatorname{Vol}\widetilde{C}\operatorname{Vol}\widetilde{C}^\circ}{(\operatorname{Vol}B_M)^2} \leq 1.$$

Since $\widetilde{C}^\circ = -\widetilde{C_i^*}$ we can rewrite this to get

$$\left(\frac{\operatorname{Vol}\widetilde{C_i^*}}{\operatorname{Vol}B_M}\right)^{1/D_M} \leq \left(\frac{\operatorname{Vol}B_M}{\operatorname{Vol}\widetilde{C}}\right)^{1/D_M}.$$

We observe that by the lower bound of Theorem 5.1 it follows that

$$\left(\frac{\operatorname{Vol}\widetilde{C_i^*}}{\operatorname{Vol}B_M}\right)^{1/D_M} \leq 2\sqrt{n(4k+2)}. \tag{8.1.1}$$

Now we apply the upper bound of Lemma 6.2 to $C$ and we get

$$\left(\frac{\operatorname{Vol}\widetilde{C_d^*}}{\operatorname{Vol}\widetilde{C_i^*}}\right)^{1/D_M} \leq \left(\frac{k!}{(n/2+k)^k}\right)^{\alpha},$$

where

$$\alpha = 1 - \left(\frac{2k-1}{n+2k-2}\right)^2.$$

The upper bound now follows by combining with (8.1.1).

## References


[1] A.I. Barvinok, *Estimating $L^\infty$ norms by $L^{2k}$ norms for functions on orbits.* Foundations of Computational Mathematics, 2 (2002), no. 4, 393-412.

[2] G. Blekherman *Convexity properties of the cone of nonnegative polynomials*, arXiv preprint math.CO/0211176 (2002), Discrete and Computational Geometry to appear.

[3] G. Blekherman *There are significantly more nonnegative polynomials than sums of squares*, arXiv preprint math.AG/0309130 (2003).

[4] M. D. Choi, T. Y. Lam, B. Reznick, *Even symmetric sextics.* Math. Z. 195 (1987), no. 4, 559-580.

[5] J. Duoandikoetxea, *Reverse Hölder inequalities for spherical harmonics.* Proc. Amer. Math. Soc. 101 (1987), no. 3, 487-491.

[6] G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities.* Reprint of the 1952 edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1988.

[7] D. Hilbert, *Über die Darstellung definiter Formen als Summe von Formenquadraten.* Math. Ann. 32, 342-350 (1888). Ges Abh. vol. 2, 415-436. Chelsea Publishing Co., New York, (1965).

[8] O. Kellogg, *On bounded polynomials in several variables.* Math. Z. 27, 1928, 55-64.

[9] M. Meyer, A. Pajor. *On the Blaschke-Santal inequality.* Arch. Math. (Basel) 55 (1990), no. 1, 82-93.

[10] J. Pach, P. Agarwal. *Combinatorial Geometry.* Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., New York, 1995.

[11] G. Pisier, *The Volume of Convex Bodies and Banach space Geometry.* Cambridge Tracts in Mathematics, 94. Cambridge University Press, Cambridge, 1989.

[12] B. Reznick, *Sums of even powers of real linear forms*, Mem. Amer. Math. Soc. 96 (1992), no. 463.

[13] B. Reznick, *Uniform denominators in Hilbert's seventeenth problem.* Math. Zeitschrift. 220 (1995), no. 1, 75–97.

[14] B. Reznick, *Some concrete aspects of Hilbert's 17th Problem.* Contemp. Math., 253 (2000), 251-272.

[15] R. Schneider, *Convex bodies: the Brunn-Minkowski theory.* Encyclopedia of Mathematics and its Applications, 44. Cambridge University Press, Cambridge, 1993.

[16] P. A. Parrilo, B. Sturmfels. *Minimizing polynomials functions.* Submitted to the DIMACS volume of the Workshop on Algorithmic and Quantitative Aspects of Real Algebraic Geometry in Mathematics and Computer Science.

[17] N. Ja. Vilenkin, *Special Functions and the Theory of Group Representations.* Translations of Mathematical Monographs, Vol. 22, American Mathematical Society (1968).

Department of Mathematics, University of Michigan,
Ann Arbor, MI 48109-1109, USA
*Email address:* gblekher@umich.edu